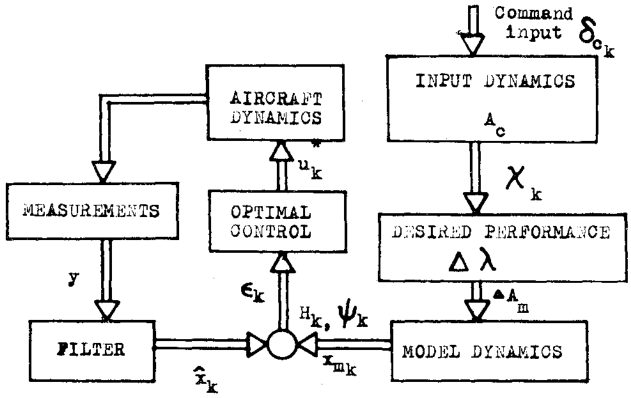


**Fig. 1 Stability-controlled control loop.**

where subscripts $R$ and $I$ denote real and imaginary parts, respectively.

For a particular aircraft the choice of the change in one of the model eigenvalues is based on its performance capabilities. Once $\Delta\lambda$ is specified, the elements of the differential matrix $\Delta A_m$ could be computed from Eq. (24).

The model transition matrix that should be used in the expression for the optimal control vector, Eq. (16), is thus

$$\Psi_k = \exp\left[A_m + \sum_{s=1}^{n} (\Delta A_m)_s\right]\Delta t$$

A block diagram for the proposed control loop is shown in Fig. 1.

## References

[1] Gelb, A. (ed.), *Applied Optimal Estimation*, MIT Press, Cambridge, Mass., 1974.

[2] Markland, C. A., "Design of Optimal and Suboptimal Stability Augmentation System," *AIAA Journal*, Vol. 8, April 1970, pp. 673-679.

[3] Gran, R., Berman, H., Rossi, M., and Rothschild, D., "Digital Flight Control for Advanced Fighter Aircraft," AIAA Paper 75-1086, Boston, Mass., Aug. 1975.

# Technical Comments

## Comment on "Simplified Methods of Predicting Aircraft Rolling Moments Due to Vortex Encounters"

Alexander H. Flax*
*Institute for Defense Analyses, Washington, D.C.*

THE simple formula derived by Barrows[1] relating the aerodynamic rolling moment according to strip theory to the rolling moment according to lifting-line theory for a wing of elliptical planform subjected to an arbitrary antisymmetric angle-of-attack distribution is a long-established and well-known result in wing theory.[2,3] Extensive applications of this formula to stability and control and to aeroelasticity have been given in the literature.[4-7] There is a similar formula[2,3] for the lift on an elliptic wing due to an arbitrary symmetric angle-of-attack distribution. These special results for the elliptic wing follow from the fact that for this planform all the coefficients of the Fourier series for spanwise circulation distribution in lifting-line theory may be determined explicitly and exactly by quadrature without solving simultaneous equations (which in principle are infinite in number). As is well known in lifting-line theory, for all planforms the total lift depends only on the first Fourier coefficient, whereas the total rolling moment depends only on the second coefficient.

The approximate formulas for rolling moments of linearly tapered wings in Ref. 1 correspond to results originally obtained by Victory and quoted in Ref. 7. As shown in Victory's results (and indicated in Barrow's work as well), the relation between rolling moment from strip theory $M_r$ and the rolling moment $M_r'$ according to lifting-line theory is

$$\frac{M_r'}{M_r} = \frac{1}{1+(c/Æ)} \tag{1}$$

where Æ is the aspect ratio and the exact value of $c$ is 4 for the elliptic wing in all cases of antisymmetric loading. For other wing planforms, $c$ is a function of both the planform and the loading so the formula is of much less general utility for these cases. For example, for linear antisymmetric angle-of-attack distributions, $c$ varies from 3.15 to 6.03 as the wing taper ratio $\lambda$ (equal to the ratio of tip chord to root chord) varies from 0 to 1.0, whereas for cubic antisymmetric twist, $c$ varies from 2.13 to 7.65 as $\lambda$ varies from 0 to 1.0. These values indicate that over the full range of $\lambda$, differences in loading may alter $c$ substantially.

There is, in fact, a development of lifting-line theory by Sears[7] in which the solution of the problem for any wing planform is made mathematically equivalent to the solution for the elliptic wing in that the spanwise circulation distribution may be given explicitly as a series of terms, each of which can be expressed independently of the others so that simultaneous equations need not be solved for each new distribution of imposed angle of attack. For tapered planforms, the Sears solution is of form

$$M_r' = qSb \sum_{n=1}^{\infty} m_n \frac{\int_0^{\pi} \alpha(\tau)\varphi_n(\tau)\mathrm{d}\tau}{1+(c_n/Æ)} \tag{2}$$

where $q$ is the dynamic pressure, $S$ is the wing area, $\alpha$ is the imposed angle of attack, and the angle $\tau$ is related to the spanwise coordinate $y$ by $y = b/2\cos\tau$, $b$ being the wing span. The functions $\varphi_n(\tau)$, of which there is a denumerably infinite set for any given wing-taper ratio, are eigenfunctions of the homogeneous linear integral equation obtained by omitting the term in geometric angle of attack from the integral equation of lifting-line theory. $c_n$ increases monotomically as $n$ increases and is, to within a multiplicative constant depending on taper ratio, the negative of the eigenvalues corresponding to $\varphi_n$. $m_n$ is effectively a rolling-moment coefficient corresponding to circulation distributed proportional to $\varphi_n(\tau)$. [In fact, in analogy to the elliptic case in which the spanwise circulation $\Gamma(\tau) = 2bU_0\Sigma A_n \sin n\tau$, for the more general case $\Gamma = 2bU_0\Sigma a_n\varphi_n(\tau)$.]

For wings with linear taper in planform, Sears has given numerical values for a representative set of taper ratios; thus $c_n$ and $m_n$ may readily be found. Unlike Eq. (1), Eq. (2) admits no simple relationship between the value of rolling moment obtained from strip theory and the value from lifting-line theory. Moreover, if $\alpha(\tau)$ is orthogonal to all values of $\varphi_n$ except, say, $\varphi_m$, then the summation contains only a single term, the $m$th, and the value of $c_m$ in the denominator

Received May 31, 1977.
Index categories: Aerodynamics; Structural Design (including Loads); Handling Qualities, Stability, and Control.
*President. AIAA Fellow.

corresponds to the $m$th mode; thus, in principle, $c_m$ may be arbitrarily large, although admittedly the corresponding distributions of lift may not be of any practical interest.

Finally, it is necessary to call attention to the fact that Barrows' quotation of reciprocity theorem for wing rolling moment for an arbitrary imposed angle-of-attack distribution is not generally true within the framework of lifting-surface theory. To paraphrase his statement in corrected form, "The rolling moment on a wing encountering an arbitrary downwash field is equal to the integral over the span of the product of the local angle of attack and the sectional lift at the corresponding spanwise station of a flat-plate wing of identical planform *in reverse flow* which is rolling at a rate $p = 2U_0/b$." (The added condition is italicized.)

The general reciprocity relations in lifting-surface theory are between wings in reverse flow. However, as pointed out in Ref. 8, the requirement for flow reversal does not apply in some cases, e.g., within the context of lifting-line theory or slender-wing theory or for wings that are symmetric fore and aft about a spanwise axis of symmetry.

## References

[1] Barrows, T. M., "Simplified Methods of Predicting Aircraft Rolling Moments Due to Vortex Encounters," *Journal of Aircraft*, Vol. 14, May 1977, pp. 434-439.

[2] Ames, J. S., "A Resumé of the Advances in Theoretical Aeronautics Made by Max M. Munk," NACA Report No. 213, March 1925.

[3] Munk, M., *Fluid Dynamics for Aircraft Designers*, Ronald Press, New York, 1929, pp. 99-102.

[4] Victory, M., "The Calculation of Aileron Reversal Speed," British Aeronautical Research Council Reports and Memoranda 2059, 1944.

[5] Duncan, W. J., *The Principles of Control and Stability of Aircraft*, Cambridge University Press, London, 1950, pp. 158-160.

[6] Flax, A. H., "High Speed Problems of Aircraft and Experimental Methods, Part I," *Aeroelasticity and Flutter, Vol. VIII*, edited by A. F. Donovan and H. R. Lawrence, Princeton University Press, Princeton, N. J., 1961, pp. 264-266.

[7] Sears, W. R., "A New Treatment of Lifting Line Theory, with Applications to Rigid and Elastic Wings," *Quarterly of Applied Mathematics*, Vol. 6, July-Sept. 1948, pp. 239-255.

[8] Flax, A. H., "General Reverse Flow and Variational Theorems in Lifting-Surface Theory," *Journal of the Aeronautical Sciences*, Vol. 19, June 1952, pp. 361-374.

# Reply by Author to A.H. Flax

Timothy M. Barrows
*Transportation Systems Center, Cambridge, Mass.*

I AM indebted to Flax for pointing out the original derivation of the simple rolling-moment relation. As might be surmised from my paper, I had a feeling this result might have been derived earlier. It certainly is a disappointment to have missed an opportunity to quote the immortal Max Munk.

The results quoted for rolling moments were not actually derived by Victory. She merely reproduced an appendix from an earlier report by Hirst and Brooke.[1] They calculated the first four even terms $A_n$ of the Prandtl-Glauert series using the method of least squares with eight control points along the span – a lot of work in those days. Referring to Eq. (31) of my paper, we see that my approximation of $c$ is $c = 4(1+\epsilon)$ for the case of $\alpha = y$ (linear antisymmetric angle of attack) which comes reasonably close to their results. I am sure Flax agrees that the proper way to get around the fact that $c$ varies whenever the angle of attack is other than linear antisymmetric is to use a reciprocal relation.

One can only guess at what motivation Flax has for bringing up the 1948 work of Sears. This work originated from a desire to solve the lifting-line equation along the lines of more classical solutions to integral equations. It did result in avoiding the need to solve simultaneous equations for each new angle-of-attack distribution. However, the price paid was too high. A series of eigenvalues and eigenfunctions must be derived for each new wing planform, and these replace the universal sines and cosines used by Glauert. The later introduction of the reciprocity theorems provided a much cleaner and simpler means of obtaining similar results. The reciprocity theorems require a single spanwise integration, whereas the Sears solution requires a whole series of integrations (except in the unusual situation mentioned by Flax).

Flax is quite correct in pointing out that to maintain generality one must include the words "in reverse flow" in the reciprocal theorem. As he so kindly points out, within the context of lifting line theory, there is no requirement for flow reversal, so the results derived in the paper are unaffected. However, this does raise the following questions: When is flow reversal required? Is it required for swept wings? The answer to the latter is "probably not" for purposes of calculating vortex encounter motions.

The upshot of my analysis is that by calculating the rolling moment in the manner prescribed by Eggleston and Diederich (Ref. 9 of my paper), the main effect of taper is contained in the roll-damping derivative; and there are only secondary effects caused by a spanwise redistribution of the loading. The same is probably true of the effects of sweep. Another reciprocal theorem of Heaslet and Spreiter (Ref. 8 of my paper), which carries no restrictions on planform, appears as follows:

> "The rolling moment per unit angular rolling velocity of flat-plate wings in steady or indicial motion is the same in forward and reverse flight."

The same statement cannot be made about pitching moments, incidentially. Using this additional theorem, we see that to first order (that is, to within the accuracy of the first term of my Eq. 38) flow reversal has no effect. Three conditions must prevail simultaneously before there is a practical effect of including flow reversal in the calculation:

1) The following wing must be substantially swept.

2) The oncoming downwash field must differ significantly from solid body rotation.

3) Predictions of rolling moment are required that are more accurate than those available from the first-order solution.

The last condition above seems very unlikely in view of the uncertainties of predicting the dynamics of a vortex encounter.

Received June 23, 1977.
Index categories: Aerodynamics; Structural Design (including Loads); Handling Qualities, Stability, and Control.

*Mechanical Engineer. Member AIAA.

## Reference

[1] Hirst, D. M. and Brooke, G. R., "On the Calculation of the Critical Reversal Speeds of Wings," *British Aeronautical Research Council Reports and Memoranda 1568*, 1933.